ON SOME ONE-DIMENSIONAL MOTIONS OF SOFT SOIL

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A class of self-similar solutions of the equations of one-dimensional motion (with plane waves) of soft soil is considered. The influence of compressibility of the soil during unloading on the motion characteristics is investigated.

1. In conformity with [1], the motions of soft soil are described by the system of equations

$$\frac{1}{\rho}\frac{d\rho}{dt}+\frac{\partial v_\alpha}{\partial x_\alpha}=0, \qquad \rho\left(\frac{dv_k}{dt}-f_k\right)+\frac{\partial p}{\partial x_k}=\frac{\partial s_{k\alpha}}{\partial x_\alpha}$$

$$G\,(e_{ik}-{}^1/_3\, e_{\alpha\alpha}\delta_{ik})=\frac{Ds_{ik}}{Dt}+\omega s_{ik}, \qquad e_{ik}=\frac{1}{2}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)$$

$$\frac{Ds_{ik}}{Dt}=\frac{ds_{ik}}{dt}-j\,(s_{i\alpha}\Omega_{k\alpha}+s_{k\alpha}\Omega_{i\alpha}), \qquad \Omega_{ik}=\frac{1}{2}\left(\frac{\partial v_i}{\partial x_k}-\frac{\partial v_k}{\partial x_i}\right)$$

$$\omega=\left(G\frac{W}{F}-\frac{d\ln\sqrt{F}}{dt}\right)e\,(I-F)\,e\left(G\frac{W}{F}-\frac{d\ln\sqrt{F}}{dt}\right) \qquad (1.1)$$

$$W={}^1/_2\, e_{\alpha\beta}s_{\alpha\beta}, \quad I={}^1/_2\, s_{\alpha\beta}s_{\alpha\beta}, \quad \rho=\rho\,(p,\ p_*), \quad F=F\,(p,\ p_*), \quad G=G\,(p,\ p_*)$$

$$\frac{dp_*}{dt}=\frac{dp}{dt}\,e\,(p-p_*)\,e\left(\frac{dp}{dt}\right), \qquad \frac{d}{dt}=\frac{\partial}{\partial t}+v_\alpha\frac{\partial}{\partial x_\alpha}$$

$$\delta_{ik}=\begin{cases}1\ (i=k),\\ 0\ (i\neq k),\end{cases} \qquad e\,(\xi)=\begin{cases}1\ (\xi\geqslant 0),\\ 0\ (\xi<0),\end{cases} \qquad j=\begin{cases}1,\\ 0\end{cases}$$

Here t is the time, x_k Cartesian coordinates, v_k velocity components, ρ density, f_k volume external force components, p hydrodynamic pressure, s_{ik} stress deviator components, j takes the value 1 when the rate of change of the stress deviator is determined according to Jaumann [1], and zero if the Jaumann supplements are ignored; $F\ (p,\ p_*)$ and $G\ (p,\ p_*)$ are functions characterizing the mechanical properties of the soil.

One-dimensional motions (with plane waves), which occur in the absence of gravity and other volume forces, are henceforth considered. All the motion parameters depend on just the one coordinate $x=x_1$, moreover

$$v_2=v_3=0, \qquad s_{22}=s_{33}=-{}^1/_2\ s_{11}, \qquad s_{12}=s_{23}=s_{31}=0$$

The system of equations describing the motion is

$$\frac{1}{\rho}\frac{d\rho}{dt}+\frac{\partial u}{\partial x}=0, \quad \rho\frac{du}{dt}+\frac{\partial X}{\partial x}=0, \quad \frac{d\Psi}{dt}+\omega\Psi=-\frac{4}{3}G\frac{\partial u}{\partial x}$$

$$W=-\Psi\frac{\partial u}{\partial x},\ I={}^3/_4\,\Psi^2,\ u=v_1,\ \Psi=-s_{11},\ X=p+\Psi,\ \frac{d}{dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x} \quad (1.2)$$

Let a pressure P_0 be applied at the initial instant $t=\tau_0$ to a free soil surface initially occupying the $x>0$ half-space, and let it then change in time according to the law

$$P=P_0\varphi\ (t/\tau_0) \qquad (1.3)$$

The function $\varphi(\xi)$ is assumed continuous and differentiable for $\xi \geqslant 1$, moreover $\varphi(1) = 1$, $\varphi'(1) < 0$, $\varphi(\xi) < 1$ for $\xi > 1$ and $\varphi(\xi) = 0$ for $\xi < 1$.

A shock travels over the ground. If its velocity of propagation is supersonic (and this is henceforth always assumed), the solution of (1.2) should satisfy the following boundary and initial conditions:

$$u = h^{\cdot}(t), \quad X = P_0\varphi(t/\tau_0) \qquad \text{for } x = h(t)$$
$$u = (1 - \rho_1/\rho)s^{\cdot}(t), \quad X = \rho_1 u s^{\cdot} \qquad \text{for } x = s(t)$$
$$h(\tau_0) = s(\tau_0) = 0 \tag{1.4}$$

Here the equations $x = h(t)$ and $x = s(t)$ are the laws of ground surface and shockwave motion, respectively, ρ_1 is the density of the soil ahead of the shock. The form of the functions $h(t)$ and $s(t)$ is unknown in advance, and their determination is a part of the problem.

It is henceforth assumed that $\rho_1 = \text{const}$, and a specific volume strain ε is introduced in place of the density ρ according to the formula

$$\varepsilon = 1 - \rho_1/\rho$$

Moreover, only such motions for which the additional conditions

$$p < p_* \quad \text{for } x < s(t), \qquad p = p_* \quad \text{for } x = s(t) \tag{1.5}$$

are satisfied will henceforth be condidered.

Great compactness in the formulation of the problem is achieved if it is written in Lagrange variables, to which we transform by means of the formulas

$$t = \tau \qquad dx = u d\tau + (1 - \varepsilon) dl \tag{1.6}$$

It is hence convenient to consider that the particles on the surface have the Lagrange coordinate $l = l_0 = \text{const}$.

Taking account of (1.5) and (1.6), Eqs. (1.2) can be represented as

$$\frac{\partial \varepsilon}{\partial \tau} + \frac{\partial u}{\partial l} = 0, \quad \rho_1 \frac{\partial u}{\partial \tau} + \frac{\partial X}{\partial l} = 0, \quad \frac{\partial \Psi}{\partial \tau} + \omega \Psi = -\frac{4}{3} G \frac{\partial u}{\partial l}(1 - \varepsilon)^{-1}, \qquad p_* = p_*(l)$$
$$W = -\Psi \frac{\partial u}{\partial l}(1 - \varepsilon)^{-1}, \quad I = {}^3/_4\, \Psi^2, \quad \varepsilon = \varepsilon(p, p_*) = 1 - \frac{\rho_1}{\rho(p, p_*)} \tag{1.7}$$

The boundary conditions (1.4) become

$$X = P_0 \varphi(\tau/\tau_0) \qquad \text{for } l = l_0$$
$$p = p_*, \; u = \varepsilon s^{\cdot}, \; X = \rho_1 u s^{\cdot} \qquad \text{for } l = l_0 + s(\tau) \tag{1.8}$$
$$s(\tau_0) = 0$$

2. The mechanical model of the soil is determined by the form of the dependences

$$\varepsilon = \varepsilon(p, p_*), \qquad F = F(p, p_*), \qquad G = G(p, p_*)$$

The specific volume strain is henceforth represented as

$$\varepsilon = (p_*/b)^n f(\pi), \quad \pi = p/p_*, \qquad n = \text{const}, \qquad b = \text{const} \tag{2.1}$$
$$f'(\pi) > 0, \quad f(1) = 1, \quad 0 < n < 1, \quad f'(1) < n, \quad 0 < f_0 < 1 \; (f_0 = f(0))$$

For $p = p_*$

$$\pi = 1, \qquad \varepsilon = \varepsilon_*(p_*) = (p_*/b)^n$$

i.e. the loading branch of the volume strain diagram is a power-series curve. Its own unloading branch corresponds to each value of p_*, but they are affine-similar curves. The parameter f_0 characterizes the residual volume strain: for $f_0 = 0$ there are no residual strains, and volume elasticity is lacking for $f_0 = 1$ (i.e. the whole volume strain is residual).

In conformity with the conceptions developed in [1], the motion of soft soil can occur in two modes: with elastoplastic shear when $\omega > 0$, and with elastic shear when $\omega = 0$.

The equation $I = F(p, p_*)$, the plasticity condition, should be satisfied in the elastoplastic mode. The function $F(p, p_*)$ is later represented as

$$F = {}^3/_4\alpha^2 p_*^2 g(\pi) \qquad (\alpha = \text{const}, \ g'(\pi) > 0, \ g(1) = 1) \tag{2.2}$$

The function $G(p, p_*)$ is analogous to the shear modulus G_e for elastic bodies, which is expressed in terms of the volume compression modulus K_e and the Poisson ratio σ of an elastic body thus

$$G_e = \frac{3}{2}\frac{1-2\sigma}{1+\sigma}K_e$$

By analogy with this formula, we can write

$$G(p, p_*) = \frac{3}{2}\varkappa(p, p_*)K(p, p_*) \quad K = (1-\varepsilon)\left(\frac{\partial\varepsilon}{\partial p}\right)^{-1} \tag{2.3}$$

Here the function $\varkappa(p, p_*)$ includes an indeterminacy in the function $G(p, p_*)$. It follows from physical considerations that the function $G(p, p_*)$ should remain finite when $\partial\varepsilon/\partial p \equiv 0$. The simplest representation for the function $\varkappa(p, p_*)$ which assures compliance with this condition, is

$$\varkappa(p, p_*) = \alpha\gamma f'(\pi) \qquad (\gamma = \text{const}) \tag{2.4}$$

Taking account of (2.4) it follows from (2.3) that

$$G(p, p_*) = {}^3/_2\ \alpha\gamma\,(p_*/\varepsilon_*)(1-\varepsilon) \tag{2.5}$$

3. By virtue of (1.2) and (2.2) it follows from the plasticity condition $I = F(p, p_*)$ that in the elastoplastic mode

$$\Psi = \alpha p_* g(\pi) \tag{3.1}$$

Therefore, in the elastic mode $\omega = 0$

$$\frac{\partial\Psi}{\partial\tau} = -\frac{4}{3}G\frac{\partial u}{\partial l}(1-\varepsilon)^{-1} \quad \text{or} \quad \frac{\partial\Psi}{\partial\tau} = 2\alpha\gamma p_* f'(\pi)\frac{\partial\pi}{\partial\tau}$$

Here (2.5) has been taken into account. Hence

$$\Psi = 2\alpha\gamma p_*(f(\pi) - f_0) \tag{3.2}$$

Here the additional condition that the stresses all vanish simultaneously has been taken into account.

The equalities

$$g(\pi) = 2\gamma(f(\pi) - f_0), \qquad g'(\pi) = 2\gamma f'(\pi) \tag{3.3}$$

which are a consequence of the assumed continuity of stresses during transition, should be satisfied in passing from the elastoplastic to the elastic mode. Equations (3.3) show that the transition occurs in each particle when the pressure p reaches the value $p_t = p_*\pi_t$, where π_t is a root of the equation

$$\Gamma'(\pi_t) = 0 \ (\Gamma(\pi) = {}^1/_2\, g(\pi)\,(f(\pi) - f_0)^{-1}) \tag{3.4}$$

For $p > p_t$ elastoplastic shear occurs in the particle, for $p < p_t$ the shear strain is elastic. Equations (3.3) define the constant $\gamma = \Gamma(\pi_t)$ in (2.4). It follows from (3.1) and (3.2) that

$$\Psi = \alpha p_*\psi(\pi), \qquad \psi(\pi) = \begin{cases} g(\pi)\ (\pi > \pi_t), \\ 2\gamma(f(\pi) - f_0)\ (\pi < \pi_t) \end{cases} \tag{3.5}$$

It is henceforth assumed that $\pi_t < 1$. Consequently

$$\psi(1) = g(1) = 1$$

It follows from (1. 8) that the equalities

$$X = P_0, \quad p = p_* = \frac{P_0}{1+\alpha}, \quad \Psi^{\cdot} = \frac{\alpha P_0}{1+\alpha}, \quad \varepsilon_0 = \left(\frac{P_0}{(1+\alpha)b}\right)^n$$

$$u = u_0 = \left(\frac{P_0\varepsilon_0}{\rho_1}\right)^{1/2}, \quad s^{\cdot}(\tau_0) = c_0 = \left(\frac{P_0}{\rho_1\varepsilon_0}\right)^{1/2}$$

are valid at the initial instant $\tau = \tau_0$.

It will be convenient to reduce (1. 7) and the boundary conditions (1. 8) to dimensionless form for the sequel, by assuming

$$\tau = \tau_0\bar{\tau}, \quad l = l_0\bar{l}, \quad u = u_0\bar{u}, \quad p = \frac{P_0}{1+\alpha}\bar{p}, \quad p_* = \frac{P_0}{1+\alpha}\bar{p}_* \tag{3.6}$$

$$\Psi^{\cdot} = \frac{\alpha P_0}{1+\alpha}\bar{\Psi}^{\cdot}, \quad X = P_0\bar{X}, \quad \varepsilon = \varepsilon_0\bar{\varepsilon}, \quad s(\tau) = l_0(S-1)$$

The constant l_0 has been indefinite up to now, It is best to remove this indefiniteness by assuming $l = c_0\tau_0$, $c_0 = s^{\cdot}(\tau_0)$. In dimensionless form Eqs. (1. 7) are (the bar is omitted over the dimensionless quantities)

$$\frac{\partial\varepsilon}{\partial\tau} + \frac{\partial u}{\partial l} = 0, \quad \frac{\partial u}{\partial\tau} + \frac{\partial X}{\partial l} = 0, \quad \varepsilon = p_*^{\,n} f(\pi), \quad p_* = p_*(l)$$

$$X = p_*\chi(\pi), \quad \pi = p/p_*, \quad \chi(\pi) = \frac{\pi + \alpha\psi(\pi)}{1+\alpha} \tag{3.7}$$

The boundary conditions (1. 8) become

$$p_* = 1, \ \chi(\pi) = \varphi(\tau) \quad \text{for } l = 1 \tag{3.8}$$

$$u = \varepsilon S^{\cdot}, \ p_* = uS^{\cdot}, \ \pi = 1 \quad \text{for } l = S(\tau)$$

$$S(1) = S^{\cdot}(1) = 1$$

4. It can be shown (this has been done earlier in [2] for $g(\pi) = \pi$) that there exists a one parameter family of functions $\chi_0(\tau, k)$ (a parameter of this family is denoted by k) such that when the function of the surface pressure $\varphi(\tau)$ from (3. 8) belongs to this family, the solution of (3. 7) which satisfies the boundary conditions (3. 8) can be expressed as

$$u = u_*(l)\vartheta(\xi), \quad p = p_*(l)\pi(\xi), \quad \xi = lS^{-1} \tag{4.1}$$

where $\vartheta(\xi)$ and $\pi(\xi)$ satisfy some ordinary differential equations. Indeed, if a solution of (3. 7) of the form (4. 1) is sought, it will result in the equations

$$\chi(\pi) = \frac{u_* S^{\cdot}}{p_* S}\xi\frac{d\vartheta}{d\xi} - \frac{p_*^{\cdot}}{lp_*}\chi'(\pi)\xi\frac{d\pi}{d\xi} \tag{4.2}$$

$$\vartheta = \frac{p_*^{\,n}S^{\cdot}}{u_*S}f'(\pi)\xi\frac{d\pi}{d\xi} - \frac{u_*^{\cdot}}{lu_*}\xi\frac{d\vartheta}{d\xi}$$

Since ϑ and π depend only on ξ, by assumption, it is necessary that the coefficients in (4. 2) depend only on ξ or be constant. Hence

$$lp_*^{\cdot}/p_* = -m = \text{const}, \qquad lu_*^{\cdot}/u_* = -k = \text{const}$$

Therefore

$$p_* = l^{-m}, \qquad u_* = l^{-k}$$

Because $l = \xi S(\tau)$ it follows that

$$\frac{u_* S^\cdot}{p_*^\cdot S} = -\frac{1}{m}\xi^{m-k+1} S^{m-k} S^\cdot$$

$$\frac{p_*{}^n S^\cdot}{u_* S} = -\frac{1}{k}\xi^{k-mn+1} S^{k-mn} S^\cdot$$

Hence

$$m = 2k/(1+n), \qquad S^\mu S^\cdot = 1$$

where we have introduced the notation

$$\mu = \frac{1-n}{1+n}k$$

Therefore

$$p_*(l) = l^{-2k/(1+n)}, \quad u_*(l) = l^{-k} \tag{4.3}$$

$$S(\tau) = ((1+\mu)\tau - \mu)^{1/(1+\mu)} \tag{4.4}$$

The system (4.2) now becomes

$$\frac{2k}{1+n}\chi(\pi) = \chi'(\pi)\xi\frac{d\pi}{d\xi} - \xi^{2+\mu}\frac{d\vartheta}{d\xi}$$

$$k\vartheta = \xi\frac{d\vartheta}{d\xi} - f'(\pi)\xi^{2+\mu}\frac{d\pi}{d\xi} \tag{4.5}$$

The solution of the system (4.5) is determined by the initial conditions

$$\pi(1) = \vartheta(1) = 1 \tag{4.6}$$

It is easy to see that the conditions of the shock (3.8) are satisfied automatically because of (4.3), (4.4) and (4.6). It is now evident that the set of functions defined by the formula

$$\chi_0(\tau) = \chi(\pi(\xi))_{\xi = S^{-1}} \tag{4.7}$$

again generates that one-parameter family (with parameter k) which we spoke of at the beginning of this Section. Solution of the system (3.7) corresponding to the functions $\varphi(\tau)$ from the family $\chi_0(\tau, k)$ have a form defined by (4.1), and are hence self-similar solutions. An investigation of the properties of these solutions reduces to studying the solutions of the system of ordinary differential equations (4.5) which satisfy the initial conditions (4.6). It is convenient to transform (4.5) by introducing a new variable in place of ξ

$$\eta = \xi^{-(1+\mu)} \tag{4.8}$$

The functions $\vartheta(\eta)$ and $\pi(\eta)$ will satisfy the ordinary differential equations

$$\frac{d\pi}{d\eta} = \frac{2k}{1+n+(1-n)k}\,\frac{\eta\chi(\pi) + {}^1/_2(1+n)\vartheta}{f'(\pi) - \eta^2\chi'(\pi)}$$

$$\frac{d\vartheta}{d\eta} = \frac{2k}{1+n+(1-n)k}\,\frac{f'(\pi)\chi(\pi) + {}^1/_2(1+n)\chi'(\pi)\vartheta}{f'(\pi) - \eta^2\chi'(\pi)} \tag{4.9}$$

$$\chi(\pi) = \frac{\pi + \alpha\psi(\pi)}{1+\alpha}, \qquad \psi(\pi) = \begin{cases} g(\pi) & (\pi > \pi_t) \\ 2\gamma(f(\pi) - f_0) & (\pi < \pi_t) \end{cases}$$

where π_t is a root of the equations

$$\Gamma'(\pi_t) = 0, \qquad \gamma = \Gamma(\pi_t), \qquad \Gamma(\pi) = {}^1/_2 g(\pi)(f(\pi) - f_0)^{-1}$$

and the initial conditions

$$\pi(1) = \vartheta(1) = 1 \tag{4.10}$$

Integration of Eqs. (4.9) with the initial conditions (4.10) for given functions $f(\pi)$, $g(\pi)$ and constants n, k can be performed numerically with the greatest of ease.

The functions $f(\pi)$ and $g(\pi)$ were given analytically as

$$f(\pi) = \frac{(1 - n/f_0)\,\pi + n/f_0}{(1-n)\,\pi + n}, \qquad g(\pi) = (1-\beta)\,\pi + \beta \qquad (\beta = \text{const}) \tag{4.11}$$

The solution was found by the Runge-Kutta method on an M-20 computer in the Moscow University Computation Center. The constant β was taken equal to 0.01, the constants n and f_0 were given values betaween 0.1 and 0.9. As is easy to see, the solutions of (4.9) possess the following asymptotic representation as $\eta \to \infty$:

$$\pi = C_1 \eta^{-2k/(1+n+(1-n)k)}, \qquad \vartheta = C_2 \eta^{-k/(1+\mu)}$$
$$\left(C_1 = \text{const}, \ C_2 = \text{const}, \ \mu = \frac{1-n}{1+n} k\right) \tag{4.12}$$

In deriving (4.12) it was assumed that the condition $\pi < \pi_t$ has been satisfied.

Equations (4.12) show that the integral curves corresponding to $k < 1$ in the (η^{-1}, π) plane arrive at the origin with an infinite slope, while the slope will be finite for $k = 1$. An infinite pressure pulse applied to the surface of the soil corresponds to this case $(k \leqslant 1)$.

For $k > 1$ the integral curves in the (η^{-1}, π) plane will touch the axis of values of the variable η^{-1} at the origin. In this case the pressure pulse will be finite. Numerical integration was carried out for the values $k \geqslant 1$; it was detected that it is necessary to select $k < k_1$ (n, f_0), because for $k \geqslant k_1$ the computer could not continue to solve the system (4.9) after some value η_1 (k, n, f_0) of the variable η. This was associated with the fact that the quantity $f'(\pi) - \eta^2\chi'(\pi)$ in the denominators of the right sides of (4.9), vanished for $\eta = \eta_1$. The functions $\pi_1(\eta)$ and $\vartheta_1(\eta)$ are found from the condition that the numerators and denominators on the right sides vanish simultaneously. This condition then reduces to the equalities

$$f'(\pi) - \eta^2\chi'(\pi) = 0, \quad \eta\chi(\pi) + {}^1/_2(1+n)\vartheta = 0$$

The above functions form a singular solution of the system (4.9). For $k \geqslant k_1$ the solution of the system (4.9) satisfying the initial conditions (4.10) intersects the singular solution at the point $\eta = \eta_1$. (An analytical investigation of this situation, although possible, is beyond the scope of this paper) (*). The functions $\pi(\eta)$ corresponding to

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*) It can be noted that $k_1 = \infty$ for $f_0 = 1$. For $f_0 = 0.1$ it has been established that $k_1 < 4$ for $n = 0.7$ and $k_1 < 2$ for $n = 0.5$.

A curious fact is disclosed by using (4.12) if a shift in the ground surface is considered which is proportional to

$$\theta_1 = \int_1^\infty \vartheta(\eta)\, d\eta$$

It is evident that

$$\theta_1 = \infty \quad \text{for} \quad k \leqslant 2/(1+n), \qquad \theta_1 < \infty \quad \text{for} \quad k > 2/(1+n)$$

If $n < 1$, then for $1 < k < 2/(1+n)$ the surface shift can turn out to be unexpectedly large although the surface pressure pulse will be finite.

$k > 1$ have a domain of negative values for $\eta > \eta_0(n, f_0, k)$ $(\pi(\eta_0) = 0)$, while the functions $\vartheta(\eta)$ are always positive. For the same n and k the negative values of π are greater in absolute value, the greater the f_0. Of fundamental interest is the determination of the surface pressure functions $\chi_0(\tau, k)$ which correspond to the solutions found for (4.9).

It follows from (4.1), (4.4) and (4.8) that

$$\eta = ((1+\mu)\tau - \mu)\, l^{-(1+\mu)} \left(\mu = \frac{1-n}{1+n} k\right) \tag{4.13}$$

Hence

$$(\eta)_{l=1} = (1+\mu)\tau - \mu \tag{4.14}$$

Therefore

$$\chi_0(\tau, k) = \chi(\pi(\eta))_{\eta=(1+\mu)\tau-\mu} \tag{4.15}$$

For $k > 1$ the domain of negative values of the functions $\chi_0(\tau, k)$ indicates the presence of a rarefaction phase during the change in the pressure applied to the surface with time.

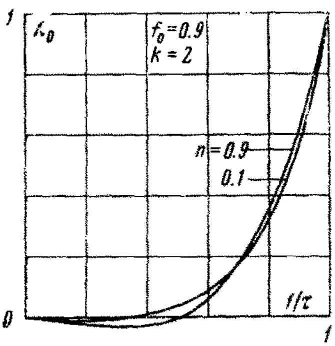

Fig. 1

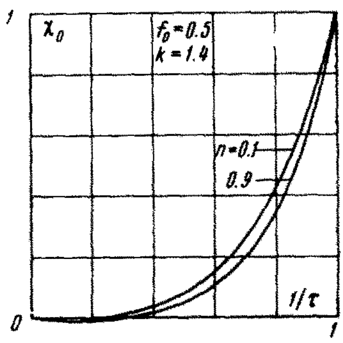

Fig. 2

Pictured in Figs. 1 and 2 are the functions $\chi_0(\tau)$ corresponding to fixed values of the parameters k and f_0 and different values of n. Only curves corresponding to the boundary values of n, equal to 0.1 and 0.9, are hence superposed on both figures. The character of the curves in Figs. 1 and 2 suggests that the functions χ_0 corresponding to constant values of k and f_0 differ slightly for different n.

This is actually so if f_0 is not too small. For the majority of soft soils $f_0 \geqslant 0.5$, hence, the reasoning expressed for such soils should be valid. It is shown in Fig. 3 that the surface pressure profiles vary as a function of k for constant n and f_0. A comparison between the curves in Fig. 3 permits the assertion that for identical laws of variation of the pressure applied to the surface of the ground the shock intensity in the soil will decrease more slowly, the smaller the residual volume strains (the shape of the loading branch of the volume strain diagram is assumed unchanged). This deduction is based on the fact that smaller k (for the identical n) corresponds to similar profiles of the function χ_0 (for example, the curve of χ_0 corresponding to $k = 1.4$, $f_0 = 0.5$ is near the curve of χ_0 corresponding to $k = 2$, $f_0 = 0.9$ so that without a coarse error they may not be distinguished), i.e. to approximately identical laws of variation of the surface pressure for small f_0. And the smaller the k, the more intense the shock, as follows from (4.4). (For times close to the initial instant the validity of this assertion can be proved completely rigorously by examining the derivative $d\chi_0/d\tau$). It is shown in Fig. 4 how the surface pressure profile varies for constant k and n as a function of f_0. Since the law of shock motion is here identical, it is shown in Fig. 4 how significant is the influence of the elasticity of the ground during unloading on its motion: the pressure profiles corresponding to lower values of f_0 are considerably less inflated than the profiles for large values of f_0, in complete conformity with the deduction on the influence of the elasticity of the soil during unloading on the shock damping law.

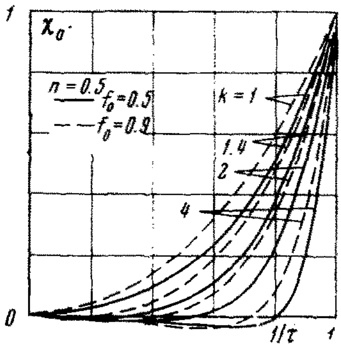

Fig. 3

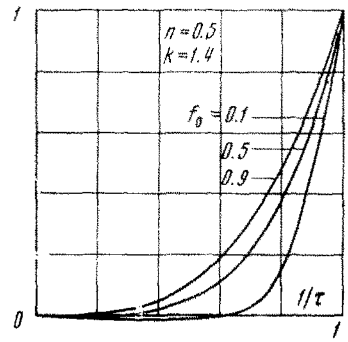

Fig. 4

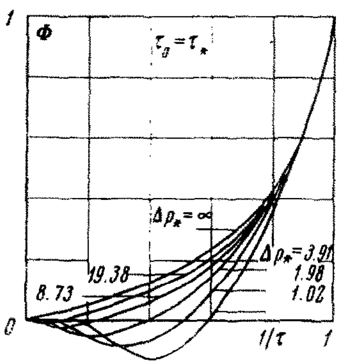

Fig. 5

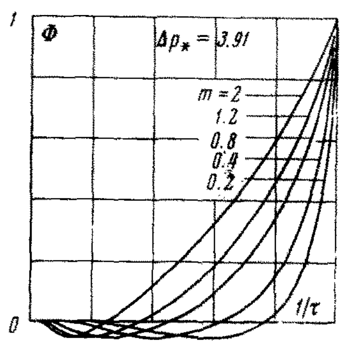

Fig. 6

It is of practical interest to know how the $\chi_0(\tau, k)$ profiles differ from the actual profiles which occur during the explosion of a charge on the surface of the ground.

Shown in Fig. 5 are pressure profiles $\Phi(\tau, \Delta p_*)$ for a "point" spherical explosion in air [3] (Δp_* is the pressure drop in an air shock wave).

The curves shown in Fig. 5 correspond to a definite selection of the time scale τ_0: for each curve $\tau_0 = \tau_*$, where τ_* is the time of arrival at the considered point of the ground by a shock of given intensity Δp_*. For another choice of τ_0 the curves in Fig. 5 are transformed in an affinely-similar manner. Fig. 6 shows how the pressure function $\Phi(\tau)$ corresonding to $\Delta p_* = 3.91$ is transformed for a different selection of the time scale τ_0 in conformity with the formula $\tau_0 = \tau_*/m$. A comparison between Figs. 3 and 6 allows the assumption that by selecting constants k and m (for given n, f_0 and Δp_*) it can be tried to achieve that the pressure profiles χ_0 and Φ be close. Excess optimism is of course out of place since, in general, the functions $\Phi(\tau)$ and $\chi_0(\tau)$ are in no way interrelated. It can be considered a lucky chance that these attempts turned out to be successful for some values of the parameters f_0 and Δp_* often encountered.

The self-similar solutions found permit an estimate of the influence of compressibility of the ground during unloading on the motion parameters even when there is no self-similarity. They can be used to verify the efficiency of the approximate methods of solution.

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THE RELATION BETWEEN MATHEMATICAL EXPECTATIONS OF STRESS AND STRAIN TENSORS IN STATISTICALLY ISOTROPIC HOMOGENEOUS ELASTIC BODIES

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The subject of this paper is the investigation of elastic solid bodies which conform to Hooke's law

$$\sigma_{ij} = c_{ijmn}\varepsilon_{mn} \tag{0.1}$$

Here the tensor of elastic moduli c_{ijmn} is considered to be a stationary random function of coordinates x_k with isotropic mathematical expectation

$$\langle c_{ijmn}\rangle = \lambda^\circ\delta_{ij}\delta_{mn} + \mu^\circ(\delta_{im}\delta_{jn} + \delta_{jm}\delta_{in}) \tag{0.2}$$

where λ° and μ° are invariant physical constants, δ_{ij} is Kronecker's tensor.

Among such bodies are for example (in the region of small deformations) polycrystals without predominant directions of anisotropy and quasi-isotropic composite bodies.

At the present time the case of macroscopically homogeneous deformation of statistically isotropic homogeneous bodies has been studied in detail in [1–4] and others. Here the relationship between the mathematical expectations of stress and strain tensors can be written in the form

$$\langle\sigma_{ij}\rangle = 2\mu\langle\varepsilon_{ij}\rangle + \lambda\langle\varepsilon_{kk}\rangle\delta_{ij} \tag{0.3}$$

where μ and λ are "effective" Lamé's constants and do not coincide with μ° and λ°. The constants mentioned can be calculated from given statistical characteristics of the stationary random tensor c_{ijmn} by solving the three-dimensional nonlinear stochastic problem. An appropriate solution in the first approximation was obtained in [1]. Most general